Group theory

Define group and write down the properties of group?

⇒ A group is a set of distinct element 'G' finite or infinite in number with a law of composition or Binary operation (addition, ~~subst~~ multiplication, matrix etc) such that the following properties are satisfied.

(i) $\forall A, B \in G$; $A \odot B \in G$ [closuer property]

(ii) There exist an identity element $E \in G$, such that

$$\forall A \in G, \ A \odot E = E \odot A = A$$

(iii) $\forall A \in G$ there exit an inverse element $B \in G$ such that $B \odot A = A \odot B = E$

(iv) $A \odot (B \odot C) = (A \odot B) \odot C$; for all $A, B, C \in G$ i.e Associative property

Define order of a Group?

The number of members in a group is known as order of Group. A group containing finite number of members is called finite group. A group containing an infinite number of members is called infinite group.

Show that $\{i, -1, -i, 1\}$ forms a group under the composition of multiplication.

⇒ (1) closuer property,

$i \cdot -1 = -i \in G$

$-1 \cdot -i = i \in G$

$-i \cdot 1 = -i \in G$

$1 \cdot i = i \in G$

$1 \cdot -1 = -1 \in G$

The closuer property hold.

(b) $i \cdot 1 = i$

$-1 \cdot 1 = -1$

$-i \cdot 1 = -i$

$1 \cdot 1 = 1$

So the identity element is

(c) $i \cdot \cancel{(\frac{1}{-i})} = i \cdot (-i) = 1$; $-i \in G$ and inverse of i

$(-1) \cdot (-1) = 1$; -1 is inverse of -1

So ; Inverse element exist for all element of G.

① $i \cdot (-1 \cdot -i) = i \cdot i = -1$

$(i \cdot -1) \cdot -i = -i \cdot -i = -1$

Associative property.

* What do you mean by Abelian group :⇒
Write down its property (" + Commuta)

The product of group members is not necessarily commutative is that.

$AB \neq BA$; $\forall A, B \in G$

if the group members holds the commutative relative

$AB = BA$; $\forall A, B \in G$

then group 'G' is called Abelian group.

Group + commutative,

② Commutative group, $AB = BA$; $\forall A, B \in G$

Prove that, a group formed by cube roots of unity is an abelian group under multiplication.

$G = \{1, \omega, \omega^2\}$

$(G, *)$

$1 \cdot \omega = \omega$ AG $1 \cdot \omega = \omega$

$\omega \cdot \omega^2 = \omega^3 = 1$ GA $\omega \cdot 1 = \omega$

$\omega^2 \cdot 1 = \omega^2$

$\omega \cdot \omega^2 = 1$, $\omega^2 \cdot \omega = 1$

commutative group satisfied.

Define cyclic group :-

A group containing a single element and its different powers is called cyclic group.

for ex :- $G = \{i, -i, 1, -1\}$

$i^1 = i$

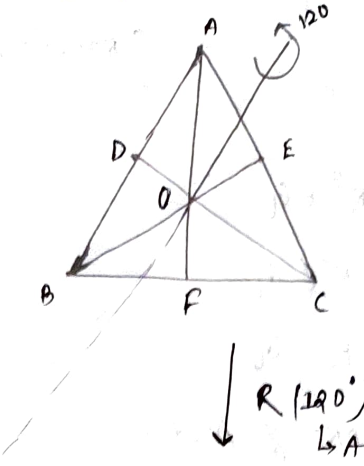
$i^2 = -1$

$i^3 = -i$

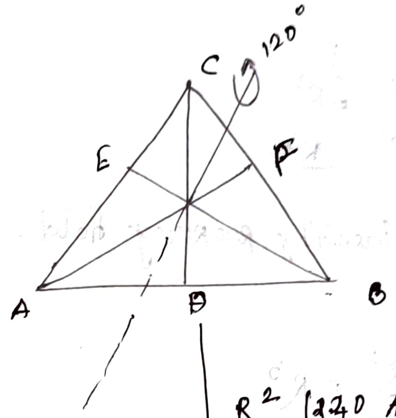
$i^4 = 1$

discuss about the rotation group formed by equilateral triangle or R^3 group.

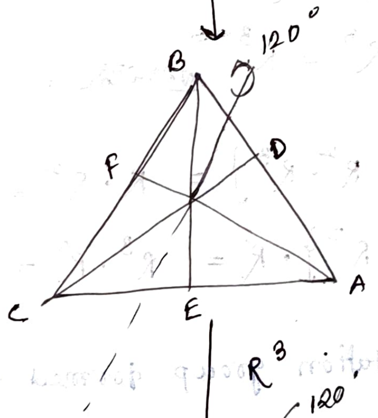
Consider a equillator triangle ΔABC, O is the centroid,



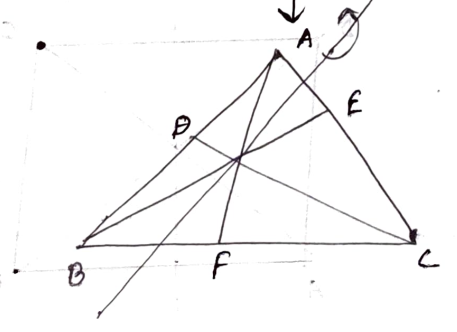
$\downarrow$ R (120°)
↳ Anti clock-wise)



$\downarrow$ R^2 (240 Anti-clock wise)



$\downarrow$ R^3



$G = (R, R^2, R^3)$ form a group of order 3.

Q. Show that $\{R, R^2, R^3\}$ form a group ? [R^3-group]

[Rotation Composition]

$G = \{R, R^2, R^3\}$

(a)

$R \cdot R^2 = R^3$, $R^3 \in G$

$R^2 \cdot R^3 = R^2$, $R^2 \in G$

$R^3 \cdot R = R$, $R \in G$

so, closure property hold.

(b)

$\underline{R} \odot R^3 = \underline{R}$

$\underline{R^2} \odot R^3 = \underline{R^2}$

$\underline{R^3} \odot R^3 = \underline{R^3}$

so, identity property hold.

(c)

$R \odot R^2 = R^3$

$R^2 \odot R = R^3$

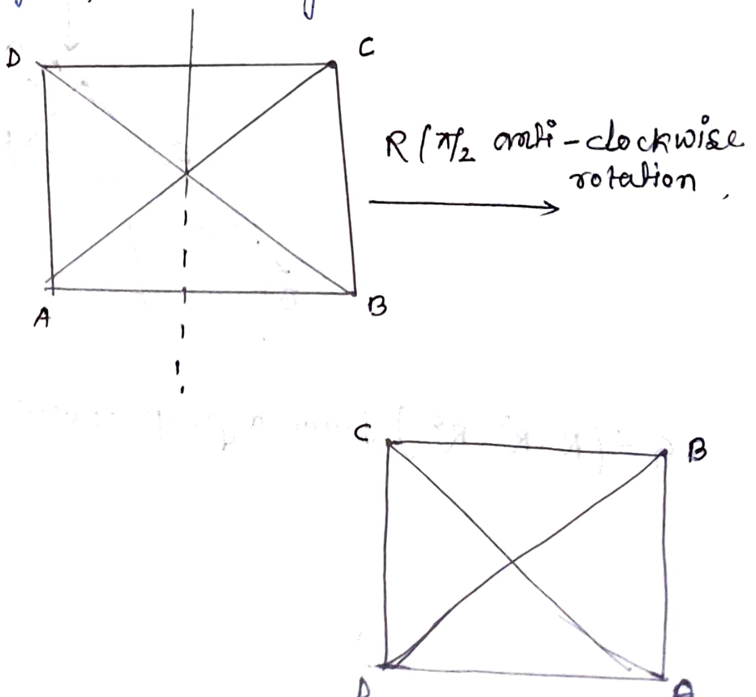
$R^3 \odot R^3 = R^3$ Inverse,

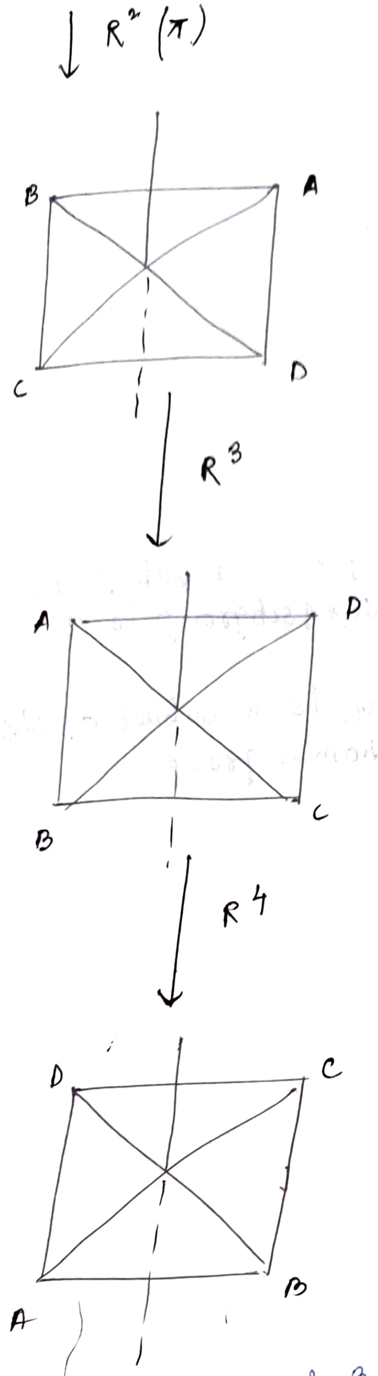
(d)

$R \cdot \{R^2 \cdot R^3\} = R \cdot R^2 = R^3$

$\{R \cdot R^2\} \cdot R^3 = R^3 \cdot R^3 = R^6 = R^3$ Associated

#. Discuss the rotation group formed by Squar or C^4 rotation group





Show that $C^4 = \{R, R^2, R^3, R^4\}$ is a group under rotation or under composition rotation.

(a)

$R \cdot R^2 = R^3 \qquad R^3 \in C^4$

$R^2 \cdot R^3 = R^5 = R \qquad R \in C^4$

$R^3 \cdot R^4 = R^3 \qquad R^3 \in C^4$

$R^4 \cdot R = R \qquad R \in C^4$

So, closuer property hold.

(b)

$R \cdot R^4 = R$

$R^2 \cdot R^4 = R^2$

$R^3 \cdot R^4 = R^3$

$R^4 \cdot R^4 = R^4$

So, Identity property hold.

$$R \cdot R^3 = R^4$$
$$R^3 \cdot R = R^4$$
$$R^4 \cdot R^4 = R^4$$

(d) $R \cdot \{R^2 \cdot R^3\} = R \cdot R^5 = R^2$

$\{R^2 \cdot R\} \cdot R^3 = R^3 \cdot R^3 = R^2$

Associative property hold.

Define sub-group and discuss properties of sub group. Show that the minimum order or index of subgroup is 2?

A Subgroup of a Group G is a subset of G's elements which themselves will from a group

With example discuss group-group multiplication :-

All the products of groups elements may be represented by a table known as group-group multiplication table.

For example: we consider the group formed by cube roots of unity under multiplication.

$$G = \{1, \omega, \omega^2\}$$

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$$G^2 = \{1, \omega, \omega^2\}$$

Q. Show that multiplication of R^3 or C^3 rotation group formed another group.

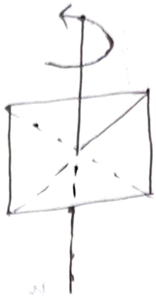
$$G = \{R, R^2, R^3\}$$

	R	R^2	R^3
R	R^2	R^3	R
R^2	R^3	R	R^2
R^3	R	R^2	R^3

$$(R^3)^2 = \{R, R^2, R^3\}$$

Define symmetry operation with an ex:

When an operation is perform on a body (the operation may be a rotation, reflection, inversion, translation or any other) And the body remains invarient then the operation is called symmetry operation.



If we performed a 90° rotation operation about a line passing through a centre of the square and perpendicular to the plane of the square on a square the square remained invarient after the operation. (Rectangle - 180°)

Discuss about the permutation in a group?

If we interchange any two or more of the object in a system of n identical objects, then the resulting configuration of the system remains unchanged.

If we consider each interchange a transformation of the system, then all such possible transformations form a group under which the system is invarient.

If there are n-number of members in a group then the possible number of permutation n!

For ex:- we consider C^3 or R^3 rotation group;

$$C^3 = \{R, R^2, R^3\}$$

All possible permutations are,

$$= \{R, R^3, R^2\}$$

$$= \{R^3, R, R^2\}$$

$$= \{R^2, R^3, R\}$$

$$= \{R^2, R, R^3\}$$

$$= \{R^3, R^2, R\}$$

Q. Discuss Isomorphism of groups.

⇒ we consider two groups G and G';

$G = \{E, A, B, C\}$ and $G' = \{E', A', B', C'\}$

Both are of same order. ~~are said~~ The groups G and G' are said to be isomorphic if there exist an unique one to one correspondence between their members such a way that the products correspon to products.

Say $E \to E'$; $A \to A'$, $B \to B'$, $C \to C'$

If $AB = C$; then $A'B' = C'$

$AB \to A'B'$

for ex: we consider C_3 rotation group and a group formed by cube roots of unity.

$C_3 = \{R, R^2, R^3\}$ and $G = \{1, \omega, \omega^2\}$

$= \{\omega, \omega^2, \omega^3\}$

$R^3 \to 1$

$R^2 \to \omega^2$

$R \to \omega$

~~$R \to \omega = E$~~

$R \to \omega$

$R^2 \to \omega^2$

$R^3 \to \omega^3$

$R \cdot R^2 = R^3$

$\omega \cdot \omega^2 = \omega^3$

If, $R \cdot R^2 = R^3$; $\omega \cdot \omega^2 = \omega^3 = 1$

$R^3 \to 1$

$R \cdot R^2 \to \omega \cdot \omega^2$

$R^2 \cdot R^3 = R^2$, ~~$\omega^2 \cdot \omega^3 = \omega^4 = \omega$~~

$\omega^2 \cdot 1 = \omega^2$

Q) Homomorphism :

Homomorphism between two groups resembles to isomorphism except that the correspondence is not required to one to one but many to one.

let us consider two groups $H = \{E, A, B, C\}$ of order 'g'.

And $G = \{E_1, E_2 \ldots, E_n, A_1, A_2, \ldots A_n; B_1, B_2 \ldots B_n; C_1, C_2 \ldots C_n\}$

of order gn.

What do you mean By kernel ? $f(a*b) = f(a) \circ f(b)$

Let f be a homomorphism of a group G to G', then the set K of ~~the~~ all those elements of G which are mapped to the identify e' of G' called the kernel of the homophisms. It is denoted by $\ker f$ or $\ker(f)$.

Define Conjugate elements of a group and classes of a group ?

Consider a relation such as $A^{-1}BA = C$ —— ①

where, A, B, C are the members of a group. When such relation exist between two members B and C then they are called Conjugate elements. The operation is called a similarity transformation of B by A.

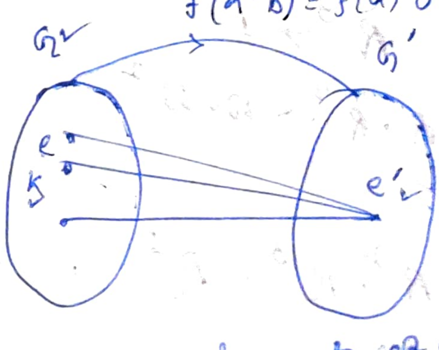
Equⁿ ① also can be represent as, $A^{-1}CA = B$

$A^{-1} \rightarrow$ inverse element of A

The members of a group which are Conjugate to each other forms a class of the group.

Q. If two elements B and C are Conjugate to a third element D, then prove that B and C also conjugate to each other.

$f(a*b) = f(a) \circ f(b)$



K of all elements of G which are